ASYMPTOTIC REPRESENTATION OF A PERIODIC SOLUTION OF THE EQUATION OF NEUTRAL TYPE WITH SMALL LAG

(ASYMPTOTICHESKOE PREDSTAVLENIE PERIODICHESKOGO RESHENIIA URAVNENIIA NEITRAL'NOGO TIPA S MALYM ZAPAZDYVANIEM)

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Consider a difference - differential equation of neutral type

$$x^{\cdot}(t) = f(t, x(t), x(t-\Delta t), x^{\cdot}(t-\Delta t)) \tag{1}$$

where $\Delta t > 0$ is a small constant lag, and assume that there exists a periodic solution with the period T of the degenerate equation

$$\chi^{\cdot}(t) = f(t, \chi(t), \chi(t), \chi^{\cdot}(t)), \qquad \chi(0) = x^{\circ} \tag{2}$$

There the following theorem is valid.

Theorem. Let the function $f = f(t, x, y, u)$ in a certain neighborhood of the degenerate solution (2) have continuous second derivatives, and satisfy in this neighborhood the condition

$$|f_u| < a < 1 \tag{3}$$

Moreover, let the degenerate solution be asymptotically stable in the first approximation, i.e.

$$f_{(x)} + f_{(y)} < -\beta, \qquad \beta > 0 \tag{4}$$

where the parenthesis around the indices indicates that the derivatives are being taken along the degenerate solution.

Then, for sufficiently small Δt, there exists a unique periodic solution with period T of equation (1).

For this solution we shall construct an asymptotic representation in powers of the small lag Δt.

Proof. Let us construct successively the functions $x_n(t)$, determining them as solutions of the equations

$$x'_{n+1}(t) = f(t, x_{n+1}(t), x_{n+1}(t-\Delta t)), x'_{n+1}(t-\Delta t)) \qquad (0 < t < T) \qquad (5)$$
$$x_{n+1}(t) = x_n(T + t) \quad \text{for } -\Delta t \leqslant t \leqslant 0$$
$$x_0(t) = \chi(t) \quad \text{for } -\Delta t \leqslant t \leqslant 0$$

From the results of the paper by Vasileva [1] it follows that the solution of equation (5) exists, if inequality (3) is fulfilled. We prove that sequence $\{x_n(t)\}$ is uniformly bounded and equicontinuous on the segment $[-\Delta t, T]$. We have

$$x' - \chi' = f(t, x_0, [x_0]), [x_0'] - f(t, \chi, \chi, \chi') = f_x^* (x_0 - \chi) + f_y^* [x_0 - \chi] + f_u^* [x' - \chi'] + R \qquad (6)$$

Here

$$|R| = |f(t, \chi, [\chi], [\chi']) - f(t, \chi, \chi, \chi')| < C_1 \Delta t$$

and we shall denote $[z] \equiv z(t - \Delta t)$; the star indicates that the arguments are taken at an intermediate point.

From [1] follows

$$|x_0 - \chi| < C\Delta t, \quad |x_0' - \chi'| < C\Delta t \qquad (-\Delta t \leqslant t \leqslant T) \qquad (7)$$

Then

$$|x_0 - \chi - [x_0 - \chi]| < C\Delta t^2, \quad |x_0' - \chi' - [x_0' - \chi']| < C\Delta t^2$$

(the second inequality follows from equation (6)); therefore equation (6) may be written

$$x_0' - \chi' = (f_{(x)} + f_{(y)})(x_0 - \chi) + f_{(u)}(x_0' - \chi') + R + O(\Delta t^2) \qquad (8)$$
$$x_0 - \chi = 0 \quad \text{for } t = 0$$

Solving this ordinary differential equation, we obtain

$$x_0 - \chi = \int_0^t [R + O(\Delta t^2)] \exp \int_s^t \frac{f_{(x)} + f_{(y)}}{1 - f_{(u)}} d\xi \, ds$$

and consequently, because of (3) and (4)

$$|x_0 - \chi| < \frac{1+a}{\beta} C_1 \Delta t + O(\Delta t^2) < \left(\frac{1+a}{\beta} C_1 + 1\right) \Delta t = C_2 \Delta t \qquad (9)$$

Denoting

$$b_1 = \max |f_x|, \qquad b_2 = \max |f_y|$$

for points pertaining to a certain neighborhood of the degenerate solution, we obtain easily from (8) and (9)

$$|x_0^{\cdot} - \chi^{\cdot}| < \left(\frac{b_1 + b_2}{1 - a} C_2 + \frac{C_1 + 1}{1 - a}\right) \Delta t \qquad (10)$$

Similarly, from [1] we have

$$|x_1 - \chi| < C\Delta t, \qquad |x_1^{\cdot} - \chi^{\cdot}| < C\Delta t$$

Then again

$$|x_1 - \chi - [x_1 - \chi]| < C\Delta t^2, \qquad |x_1^{\cdot} - \chi^{\cdot} - [x_1^{\cdot} - \chi^{\cdot}]| < C\Delta t^2$$

Therefore

$$x_1^{\cdot} - \chi^{\cdot} = (f_{(x)} + f_{(u)})(x_1 - \chi) + f_{(u)}(x_1^{\cdot} - \chi^{\cdot}) + R + O(\Delta t^2)$$

From this, using (5) and (9), we obtain

$$|x_1 - \chi| < C_2\Delta t\, e^{-\gamma t} + C_2\Delta t, \qquad \gamma = \frac{\beta}{1 + a}$$

$$|x_1^{\cdot} - \chi^{\cdot}| < \frac{b_1 + b_2}{1 - a} C_2\Delta t \cdot (e^{-\gamma t} + 1) + \frac{C_1 + 1}{1 - a} \Delta t$$

Reasoning analogously we obtain

$$|x_{n+1} - \chi| < C_2\Delta t\, [e^{-\gamma t}(1 + a_1 + a_1^2 + \ldots + a_1^n) + 1] <$$

$$< C_2\Delta t \left[\frac{1}{1 - a_1} + 1\right] = \frac{2 - a_1}{1 - a_1} C_2\Delta t \qquad (a_1 = e^{-\gamma T} < 1) \qquad (11)$$

$$|x^{\cdot}_{n+1} - \chi^{\cdot}| < \frac{(b_1 + b_2)(2 - a_1)}{(1 - a)(1 - a_1)} C_2\Delta t + \frac{C_1 + 1}{1 - a} \Delta t \qquad (12)$$

From this follows the uniform boundedness and equicontinuity of sequence $\{x_n(t)\}$ on the segment $-\Delta t \leqslant t \leqslant T$.

Let us prove that on this segment sequence $\{x_n(t)\}$ and the sequence of the derivatives $\{x_n^{\cdot}(t)\}$ are uniformly convergent. The limit function will evidently give the solution of the problem posed.

Thus, let us consider the difference $x_{n+1} - x_n$. It satisfies the equation

$$x^{\cdot}_{n+1} - x_n^{\cdot} = f_x^* (x_{n+1} - x_n) + f_u^* [x_{n+1} - x_n] + f_u^* [x^{\cdot}_{n+1} - x_n^{\cdot}]$$

As in the proof of the uniform boundedness of the sequence, we pass to the ordinary differential equation, estimating first the differences $|x_{n+1} - x_n - [x_{n+1} - x_n]|$, $|x_{n+1}^{\cdot} - x^{\cdot}_n - [x^{\cdot}_{n+1} - x_n^{\cdot}]|$.

As a result we obtain

$$x^{\cdot}_{n+1} - x_n = (f_x^* + f_y^*)(x_{n+1} - x_n) + f_u^*(x^{\cdot}_{n+1} - x^{\cdot}_n) + O(R_n)$$

or

$$x^{\cdot}_{n+1} - x^{\cdot}_n = \frac{f_x^* + f_y^*}{1 - f_u^*}(x_{n+1} - x_n) + O(R_n) \tag{13}$$

$$|R_n| < C\Delta t(\xi_n + \xi_{n-1} + \eta_n + \eta_{n-1}), \qquad \xi_n = \max|x_{n+1} - x_n|$$

$$\eta_n = \sup|x^{\cdot}_{n+1} - x^{\cdot}_n| \tag{14}$$

$$x_{n+1}(t) - x_n(t) = (x_{n+1}(0) - x_n(0)) \exp \int_0^t \frac{f_x^* + f_y^*}{1 - f_u^*} ds + O(R_n) \tag{15}$$

From (15) it is easy to obtain

$$|x_{n+1}(T) - x_n(T)| < a_1|x_{n+1}(0) - x_n(0)| + |O(R_n)|$$

But by construction

$$x_{n+1}(T) - x_n(T) = x_{n+2}(0) - x_{n+1}(0)$$

therefore the inequality takes on the form

$$|x_{n+2}(0) - x_{n+1}(0)| < a_1|x_{n+1}(0) - x_n(0)| + |O(R_n)| \tag{16}$$

From the formula analogous to (15) it is easy to conclude for the difference $x_{n+2} - x_{n-1}$

$$\xi_{n+1} < |x_{n+1}(0) - x_{n+1}(0)| + O(R_{n+1})| \tag{17}$$

On the other hand, by definition

$$|x_{n+1}(0) - x_n(0)| \leqslant \xi_n \tag{18}$$

By means of (16), (17), (18), (13) and (14) we obtain the recurrence formulas

$$\xi_{n+2} < a_1\xi_{n+1} + C\Delta t(\xi_{n+1} + \xi_n + \eta_{n+1} + \eta_n) \tag{19}$$

$$\eta_{n+2} < \frac{b_1 + b_2}{1 - a}\xi_{n+2} + C\Delta t(\xi_{n+1} + \xi_n + \eta_{n+1} + \eta_n)$$

Let $\alpha_0 = \max(\xi_1, \xi_2, \eta_1, \eta_2)$. Because of estimates (11) and (12) we have $\alpha_0 < C\Delta t$. We prove that for $n > 0$ the formula holds

$$\xi_{n+2} < (a_1 + \varepsilon)^n \alpha_0 \qquad (\varepsilon > 0) \tag{20}$$

where ε is arbitrarily small for $\Delta t \to 0$. We use the method of induction. From (19) we have

$$\xi_3 < a_1 a_0 + C\Delta t\, 4\alpha_0 = (a_1 + 4C\Delta t)\,\alpha_0 < (a_1 + \varepsilon)\,\alpha_0$$

if

$$\varepsilon \geqslant 4C\Delta t \tag{21}$$

If together with (21)

$$\varepsilon \geqslant C\Delta t\,\left(1 + \frac{b_1 + b_2}{1-a} + \frac{2}{a_1} + \frac{4C\Delta t}{a_1}\right) \tag{22}$$

is also fulfilled, then

$$\xi_4 < a_1 (a_1 + \varepsilon)\,\alpha_0 + C\Delta t\,(2\alpha_0 + (a_1 + \varepsilon)\,\alpha_0 + \frac{b_1 + b_2}{1-a}\,(a_1 + \varepsilon)\,\alpha_0 + 4C\Delta t\alpha_0) <$$
$$< (a_1 + \varepsilon)\,\left(a_1 + C\Delta t + \frac{b_1 + b_2}{1-a}\,C\Delta t + \frac{2C\Delta t}{a_1 + \varepsilon} + \frac{4C^2\Delta t^2}{a_1 + \varepsilon}\right)\alpha_0 < (a_1 + \varepsilon)^2\,\alpha_0$$

Let moreover ε satisfy also the estimate

$$\varepsilon \geqslant C\Delta t\left(1 + \frac{b_1 + b_2}{1-a}\right)\left(1 + \frac{2}{a_1}\right) \tag{23}$$

Assuming now that for $n \leqslant m$ inequality (20) is true, we prove it for $n = m + 1$. From (19) we have

$$\xi_{m+3} < a_1 (a_1 + \varepsilon)^m\,\alpha_0 + C\Delta t\Big[(a_1 + \varepsilon)^m\,\alpha_0 + (a_1 + \varepsilon)^{m-1}\alpha_0 +$$
$$+ \frac{b_1+b_2}{1-a}\,(a_1 + \varepsilon)^m\,\alpha_0 + \frac{b_1 + b_2}{1-a}\,(a_1 + \varepsilon)^{m-1}\,\alpha_0 + C\Delta t\,[(a_1 + \varepsilon)^{m-1}\,\alpha_0 +$$
$$+ (a_1 + \varepsilon)^{m-2}\,\alpha_0 + \frac{b_1 + b_2}{1-a}\,(a_1 + \varepsilon)^{m-1}\,\alpha_0 + \frac{b_1 + b_2}{1-a}\,(a_1 + \varepsilon)^{m-2}\,\alpha_0 +$$
$$+ C\Delta t\Big[(a_1 + \varepsilon)^{m-2}\,\alpha_0 + (a_1 + \varepsilon)^{m-3}\,\alpha_0 + \frac{b_1 + b_2}{1-a}\,(a_1 + \varepsilon)^{m-2}\,\alpha_0 +$$
$$+ \frac{b_1 + b_2}{1+a}(a_1 + \varepsilon)^{m-3}\alpha_0 + \ldots\Big]\ldots\Big]\ldots\Big] < \alpha_0\,(a_1 + \varepsilon)^m\Big[a_1 + C\Delta t\left(1 + \frac{b_1 + b_2}{1-a}\right) +$$
$$+ \frac{C\Delta t}{a_1 + \varepsilon}\left(1 + \frac{b_1 + b_2}{1-a}\right)(1 + C\Delta t) + \left(\frac{C\Delta t}{a_1 + \varepsilon}\right)^2\left(1 + \frac{b_1 + b_2}{1-a}\right)(1 + C\Delta t) + \ldots\Big] <$$
$$< \alpha_0\,(a_1 + \varepsilon)^m\Big[a_1 + C\Delta t\left(1 + \frac{b_1 + b_2}{1-a}\right) + \left(1 + \frac{b_1 + b_2}{1-a}\right)(1 + C\Delta t)\,\times$$
$$\times\,\left(\frac{C\Delta t}{a_1 + \varepsilon - C\Delta t}\right)\Big] = \alpha_0\,(a_1 + \varepsilon)^m\Big[a_1 + C\Delta t\left(1 + \frac{b_1 + b_2}{1-a}\right)\left(1 + \frac{C\Delta t + 1}{a_1 + \varepsilon - C\Delta t}\right)\Big]$$

Since estimate (23) is fulfilled, we have

$$\xi_{m+3} < (a_1 + \varepsilon)^{m+1}\alpha_0$$

which was to be proved.

Since $a_1 + \varepsilon < 1$, from (20) and (19) follows the uniform convergence of sequences $\{x_n(t)\}$ and $\{x_n^{\cdot}(t)\}$ on the entire interval $-\Delta t \leqslant t \leqslant T$.

Thus, the existence of a periodic solution of equation (1) is proved.

We prove that in a small neighborhood of the degenerate solution there does not exist another periodic solution of equation (1) different from the constructed one.

Let another periodic solution x_1 of equation (1) exist. Then, repeating for the difference $x - x_1$ the reasoning carried out for the proof of the convergence of sequence $\{x_n(t)\}$, we obtain

$$\max_{[nT,(n+1)T]} |x - x_1| < (a_1 + \sigma)^{n-2} \max_{[0,T]} |x - x_1|$$

where $\sigma > 0$ is small, as is the selected neighborhood of the degenerate solution.

Since x and x_1 are periodic

$$\max_{[nT,(n+1)T]} |x - x_1| = \max_{[0,T]} |x - x_1|$$

and since in a small neighborhood of the degenerate solution $a_1 + \sigma < 1$, $|x - x_1| \equiv 0$, as was to be proved.

The theorem is now proved. We pass to the question of construction of the asymptotic representation.

By construction of the periodic solution and because of inequalities (11) and (12), we obtain the formula for the zero approximation

$$X_0 = x_0 \equiv \chi, \qquad |x - X_0| < C\Delta t, \qquad |x^\cdot - X_0^\cdot| < C\Delta t \tag{24}$$

Consider the following series whose partial sums give the necessary asymptotic formulas:

$$\sum_{k=0}^{\infty} \Delta t^k x_k(t) \tag{25}$$

Substituting this series into equation (1), expanding the right-hand side by powers of Δt, and equating the coefficients of equal powers of Δt, we obtain equations for the terms of series (25). For example

$$x_0^\cdot = f(t, x_0(t), x_0(t), x_0^\cdot(t)), \quad x_1^\cdot = f_{(x)} x_1 + f_{(y)}(x_1 - x_0^\cdot) + f_{(u)}(x_1^\cdot - x_0^{\cdot\cdot})$$
$$x_2^\cdot(t) = f_{(x)} x_2 + f_{(y)}\left(x_2 + \frac{x_0^{\cdot\cdot}}{2!} - x_1^\cdot\right) + f_{(u)}\left(x_2^\cdot + \frac{x_0^{\cdot\cdot\cdot}}{2!} - x_1^\cdot\right) + f_{(xx)}\frac{x_1^2}{2!} +$$
$$+ f_{(yy)}\left(\frac{x_1^2}{2!} + \frac{x_0^{\cdot 2}}{2!} - x_1 x_0^\cdot\right) + f_{(uu)}(\ldots) + f_{(xy)}(\ldots) + f_{(xu)}(\ldots) + f_{(yu)}(\ldots)$$

However, to determine the terms of series (25), it is necessary to give the initial conditions. The initial condition for the zeroth term $x_0(0) = x^0$ is known, therefore $x_0(t)$ will be found. We transcribe now the equation for the functions $x_k (k = 1, \ldots, n)$, separating in it the terms containing x_k, and denote the remainder by $D_k(t)$

$$x_k^{\cdot}(t) = (f_{(x)} + f_{(y)})\, x_k + f_{(u)}\, x_k^{\cdot} + D_k(t)$$

Or, solving for the derivative, we obtain

$$x_k^{\cdot} = Fx_k + B_k, \qquad F = (f_{(x)} + f_{(y)})(1 - f_{(u)})^{-1}, \quad B_k = D_k(1 - f_{(u)})^{-1} \quad (26)$$

We remark that the remainder $D_k(t)$ contains functions $x_i(t)$ for $i < k$. Assume that all $x_i(t)$ for $i < k$ are found and that all the functions $x_i(t)$ $(i < k)$ are periodic with period T, we shall determine then $x_k(0)$ from the condition of periodicity of the function $x_k(t)$.

Solving equation (26) we obtain

$$x_k(T) = x_k(0) \exp \int_0^T F(t)\, dt + \int_0^T B_k(s) \exp \int_s^T F(\xi)\, d\xi\, ds$$

But by the condition of periodicity $x_k(T) = x_k(0)$. Therefore we have finally

$$x_k(0) = \left(\int_0^T B_k(s) \exp \int_s^T F(\xi)\, d\xi\, ds \right) \left(1 - \exp \int_0^T F(t)\, dt \right)^{-1} \quad (27)$$

Thus, knowing $x_i(t)$ for $i < k$, we determine also $x_k(t)$. Since for unity the assertion is true, using the method of induction, one can find all $x_k(t)$ for $k = 0, 1, \ldots, n$. We remark that all functions $x_k(t)$ are periodic with the period T.

We prove now that if the initial conditions are determined by formula (27), then the function

$$X_n = \sum_{k=0}^{n} \Delta t^k x_k \quad (28)$$

differs from the periodic solution of equation (1) by a magnitude of the order $O(\Delta t^{n+1})$.

As was already established, $\Delta_0 = x - x_0 = O(\Delta t)$ and $\Delta_0^{\cdot} = x^{\cdot} - x_0^{\cdot} = O(\Delta t)$.

Let us separate now in the equation for Δ_0 the terms with accuracy to $O(\Delta t^2)$. We have

$$\Delta_0^{\cdot} = f(t, \Delta_0 + x_0, [\Delta_0 + x_0], [\Delta_0^{\cdot} + x_0^{\cdot}]) - f(t, x_0, x_0, x_0^{\cdot}) = f_x^* \Delta_0 + f_y^* [\Delta_0] + f_u^* [\Delta_0^{\cdot}] + f(t, x_0, [x_0], [x_0^{\cdot}]) - f(t, x_0, x_0, x_0^{\cdot}) = f_{(x)}\Delta_0 + f_{(y)}\Delta_0 + f_{(u)}\Delta_0^{\cdot} + D_1(t)\,\Delta t + O(\Delta t^2)$$

or

$$\Delta_0^{\cdot} = F\Delta_0 + B_1\Delta t + O(\Delta t^2) \tag{29}$$

Because of the periodicity of the functions x and x_0, their difference $\Delta_0(t)$ is also periodic with the period T. Solving equation (29) and using equality $\Delta_0(0) = \Delta_0(T)$, we find

$$\Delta_0(0) = \Delta t\left(\int_0^T B_1(s)\exp\int_s^T F(\xi)\,d\xi\,ds\right)\left(1 - \exp\int_0^T F(t)\,dt\right)^{-1} + O(\Delta t^2) \tag{30}$$

Comparing formulas (26) and (29), (27) and (30) for $k = 1$, we find

$$|x - (x_0 + \Delta t x_1)| < C\Delta t^2, \qquad |x^{\cdot} - (x_0^{\cdot} + \Delta t x_1^{\cdot})| < C\Delta t^2$$

Thus, let for $k \leqslant n - 1$ the estimate

$$|x - X_k| < C\Delta t^{k+1}, \qquad |x^{\cdot} - X_k^{\cdot}| < C\Delta t^{k+1}$$

be proved. We show that it is true also for $k = n$. Indeed, proceeding as in the proof of the first approximation, separating in the equation for the difference $\Delta_{n-1} = x - X_{n-1}$ terms with accuracy to $O(\Delta t^{n+1})$ we find

$$\Delta_{n-1}^{\cdot} = F\Delta_{n-1} + \Delta t^n B_n + O(\Delta t^{n+1}) \tag{31}$$

Since $\Delta_{n-1}(0) = \Delta_{n-1}(T)$, we find from (31)

$$\Delta_{n-1}(0) = \Delta t^n\left(\int_0^T B_n(s)\exp\int_s^T F(\xi)\,d\xi\,ds\right)\left(1 - \exp\int_0^T F(t)\,dt\right)^{-1} + O(\Delta t^{n+1}) \tag{32}$$

Comparing (31) and (26), (32) and (27), we find

$$|x - X_n| < C\Delta t^{n+1}, \quad |x^{\cdot} - X_n^{\cdot}| < C\Delta t^{n+1} \tag{33}$$

which was to be proved. From the above it is clear that for estimate (33) to be fulfilled, it is sufficient that the function f be $n + 1$ times continuously differentiable in a certain neighborhood of the degenerate solution ($n \geqslant 1$).

We remark that all the results obtained hold for the case of a system of equations of neutral type, whereby condition (3) is replaced by the following: the eigenvalues $\lambda_k(t)$ of the matrix $f_{(u)}$ do not exceed unity in the modulus.

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